STA 712 Homework 5

Due: Friday, October 28, 12:00pm (noon) on Canvas.

Instructions: Submit your work as a single PDF. For this assignment, you may include written work by scanning it and incorporating it into the PDF. Include all R code needed to reproduce your results in your submission.

Cumulants and cumulant generating functions

1. Let Y be a random variable, and recall that the *moment generating function* (MGF) of Y is given by
$$M(t) = \mathbb{E}[e^{tY}].$$
We call M the moment generating function because
$$\frac{d^k}{dt^k} M(t)\Big|_{t=0} = \mathbb{E}[Y^k].$$
We also define the *cumulant generating function* (CGF): $C(t) = \log M(t)$.

   (a) Show that
$$\frac{d}{dt} C(t)\Big|_{t=0} = \mathbb{E}[Y].$$

   (b) Show that
$$\frac{d^2}{dt^2} C(t)\Big|_{t=0} = Var(Y).$$

GLMs and the canonical link function

2. Suppose we are interested in modeling a response variable Y_i, given explanatory variables X_i. We use the generalized linear model
$$Y_i \sim EDM(\mu_i, \phi)$$
$$g(\mu_i) = \beta^T X_i,$$
where $f(Y_i; \theta, \phi) = a(Y_i, \phi) \exp\left\{ \dfrac{Y_i \theta_i - \kappa(\theta_i)}{\phi} \right\}$, and g is the canonical link function (that is, $g(\mu_i) = \theta_i$, the canonical parameter). One reason the canonical link function is nice is that it makes Fisher scoring nice.

   (a) Show that $U(\beta) = \dfrac{X^T(Y - \boldsymbol{\mu})}{\phi}$, where X is the design matrix, $Y = (Y_1, ..., Y_n)^T$, and $\boldsymbol{\mu} = (\mu_1, ..., \mu_n)^T$.

   (b) Show that $\mathcal{I}(\beta) = \dfrac{X^T V X}{\phi}$, where $V = \text{diag}(V(\mu_1), ..., V(\mu_n))$, and $V(\mu_i) = Var(Y_i)/\phi$.

Practice with exponential dispersion models

3. Suppose $Y \sim Gamma(\alpha, \beta)$, with shape $\alpha > 0$ and scale $\beta > 0$. The density function for Y is given by

$$f(y; \alpha, \beta) = \frac{y^{\alpha-1}}{\Gamma(\alpha)\beta^{\alpha}} \exp\{-y/\beta\}.$$

   (a) Show that the gamma distribution is an EDM by identifying θ, $\kappa(\theta)$, and ϕ.

   (b) Find $\mu = \mathbb{E}[Y]$ as a function of α and β by using the fact that $\mu = \frac{\partial}{\partial\theta}\kappa(\theta)$.

   (c) Using (b), what is the canonical link function for the gamma distribution?

   (d) Using the fact that $Var(Y) = \phi \cdot \frac{\partial \mu}{\partial \theta} = \phi \cdot V(\mu)$, find $V(\mu)$ as a function of μ, and find $Var(Y)$ as a function of α and β.

   (e) Show that the unit deviance for the gamma distribution is

$$d(y, \mu) = 2\left(-\log\frac{y}{\mu} + \frac{y-\mu}{\mu}\right).$$

   (f) Write the density function for Y in dispersion model form.

4. A very cool property of EDMs is that the mean-variance relationship encoded by $V(\mu)$ uniquely determines the EDM. For example, a normal distribution is the only EDM for which $V(\mu) = 1$, a Poisson distribution is the only EDM for which $V(\mu) = \mu$, etc. This means that, given a desired mean-variance relationship, we can work backwards to figure out what the EDM should be!

   Suppose we are told that $Y \sim EDM(\mu, \phi)$, and we know that $V(\mu) = \mu^3$. In this problem, we will derive the corresponding EDM for Y.

   (a) Using the fact that $V(\mu) = \frac{\partial\mu}{\partial\theta}$, find θ as a function of μ. *Hint: recall from calculus that* $\frac{\partial\theta}{\partial\mu} = \frac{1}{\partial\mu/\partial\theta} = \mu^{-3}$.

   (b) Using the fact that $\mu = \frac{\partial\kappa(\theta)}{\partial\theta}$, show that $\kappa(\theta) = -\sqrt{-2\theta} = -\frac{1}{\mu}$.

   (c) Conclude that $f(y; \mu, \phi) = a(y, \phi)\exp\left\{\frac{-y/(2\mu^2) + 1/\mu}{\phi}\right\}$. Then rearrange to show that

$$f(y; \mu, \phi) = b(y, \phi)\exp\left\{-\frac{1}{2\phi}\frac{(y-\mu)^2}{y\mu^2}\right\}.$$

   The density function of an inverse Gaussian distribution is given by

$$f(y; \mu, \phi) = (2\pi y^3 \phi)^{-1/2}\exp\left\{-\frac{1}{2\phi}\frac{(y-\mu)^2}{y\mu^2}\right\},$$

   so we have shown that if $V(\mu) = \mu^3$, then Y must have an inverse Gaussian distribution!