Exponential dispersion models

Last time: Poisson regression

Y_i (a count variable $\rightarrow$ values $0, 1, 2, \ldots$)

$$Y_i \sim \text{Poisson}(\lambda_i) \quad \text{(random component)}$$

$$g(\lambda_i) = \beta^T X_i \quad \text{(systematic component)}$$

<u>canonical link</u>: $g(\lambda_i) = \log(\lambda_i)$

$$f(y; \lambda) = \frac{e^{-\lambda} \lambda^y}{y!} = \frac{\exp\{y \log \lambda - \lambda\}}{y!}$$

$$= a(y, \phi) \exp\left\{ \frac{y\theta - \kappa(\theta)}{\phi} \right\}$$

<u>Exponential dispersion model</u> (EDM)

$$a(y, \phi) = \frac{1}{y!}, \quad \theta = \log \lambda, \quad \kappa(\theta) = \lambda, \quad \phi = 1$$

$\mu = \beta^T x_i$

Examples of EDMs

$$f(y;\theta,\phi) = a(y,\phi)\exp\left\{\frac{y\theta - \kappa(\theta)}{\phi}\right\}$$

Normal: $Y \sim N(\mu, \sigma^2)$ (σ^2 is known)

$$f(y;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{1}{2\sigma^2}(y-\mu)^2\right\}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{\frac{y\mu - (\mu^2/2)}{\sigma^2} - \frac{y^2}{2\sigma^2}\right\}$$

$\phi = \sigma^2$, $\theta = \mu$, $\kappa(\theta) = \frac{\mu^2}{2} = \frac{\theta^2}{2}$,

$$a(y,\phi) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{y^2}{2\sigma^2}\right\}$$

<u>Bernoulli</u> : $Y \sim \text{Bernoulli}(p)$

$$f(y;p) = p^y (1-p)^{1-y}$$

$$= \exp\{ y \log p + (1-y)\log(1-p) \}$$

$$= \exp\{ y \log\left(\frac{p}{1-p}\right) + \log(1-p) \}$$

$\phi = 1$, $a(y,\phi) = 1$, $\theta = \log\left(\frac{p}{1-p}\right)$, $K(\theta) = -\log(1-p)$

θ is always a function of $\mathbb{E}[Y]$

EDM components

EDM: $f(y;\theta,\phi) = a(y,\phi)\exp\left\{\frac{y\theta - \kappa(\theta)}{\phi}\right\}$

- $a(y,\phi)$ is a normalizing function
- θ canonical parameter: function of $E[Y]$

  Let $\mu = E[Y]$, $\theta = g(\mu)$ (g is a monotonically increasing function)
- ϕ dispersion parameter, related to the variance

  $Var(Y) = \phi \cdot V(\mu)$

  Ex: $N(\mu, \sigma^2)$ $\quad Var(Y) = \underbrace{\sigma^2}_{\phi} \cdot \underbrace{1}_{V(\mu)}$

  Bernoulli(p): $\mu = p$ $\quad Var(Y) = \underbrace{1}_{\phi} \cdot \underbrace{\mu(1-\mu)}_{V(\mu)}$

  Poisson(λ): $\mu = \lambda$, $Var(Y) = \underbrace{1}_{\phi} \cdot \underbrace{\mu}_{V(\mu)}$

Cumulants and the cumulant generating function

$\kappa(\theta)$ cumulant function

- derivatives of $\kappa(\theta)$ give mean & variance of Y

Recall MGF: $M(t) = \mathbb{E}[e^{tY}]$

$$\mathbb{E}[Y^r] = \frac{d^r}{dt^r} M(t)\Big|_{t=0}$$

Cumulant generating function (CGF):

$$C(t) = \log M(t) = \log \mathbb{E}[e^{tY}]$$

$$\frac{d}{dt} C(t)\Big|_{t=0} = \mathbb{E}[Y] \quad \text{(HW)}$$

$$\frac{d^2}{dt^2} C(t)\Big|_{t=0} = \mathrm{Var}(Y) \quad \text{(HW)}$$

For an EDM:

$$M(t) = \mathbb{E}[e^{tY}] = \int_y e^{ty}\, a(y,\phi) \exp\left\{ \frac{y\theta - \kappa(\theta)}{\phi} \right\} dy$$

$$= \int_y a(y,\phi) \exp\left\{ \frac{y\theta - \kappa(\theta) + ty\phi}{\phi} \right\} dy$$

$$\theta^* = \theta + t\phi$$

$$\Rightarrow M(t) = \int_y a(y,\phi) \exp\left\{ \frac{y\theta^* - \kappa(\theta^*) + \kappa(\theta^*) - \kappa(\theta)}{\phi} \right\} dy$$

$$= \exp\left\{ \frac{\kappa(\theta^*) - \kappa(\theta)}{\phi} \right\} \underbrace{\int_y a(y,\phi) \exp\left\{ \frac{y\theta^* - \kappa(\theta^*)}{\phi} \right\} dy}_{=1}$$

$$= \exp\left\{ \frac{\kappa(\theta^*) - \kappa(\theta)}{\phi} \right\} \Rightarrow C(t) = \frac{\kappa(\theta + t\phi) - \kappa(\theta)}{\phi}$$

$$C(t) = \frac{\kappa(\theta + t\phi) - \kappa(\theta)}{\phi}$$

$$\frac{d}{dt} C(t) = \kappa'(\theta + t\phi) \Rightarrow \frac{\partial}{\partial t} C(t) \Big|_{t=0} = \kappa'(\theta) = \frac{\partial \kappa(\theta)}{\partial \theta}$$

$$\Rightarrow \mu = \mathbb{E}[Y] = \frac{\partial \kappa(\theta)}{\partial \theta}$$

$$\frac{\partial^2}{\partial t^2} C(t) \Big|_{t=0} = \phi \cdot \frac{\partial^2 \kappa(\theta)}{\partial \theta^2}$$

$$\Rightarrow Var(Y) = \phi \cdot \frac{\partial^2}{\partial \theta^2} \kappa(\theta)$$

$$= \phi \cdot \frac{\partial}{\partial \theta} \left(\frac{\partial}{\partial \theta} \kappa(\theta) \right)$$

$$= \phi \cdot \underbrace{\frac{\partial}{\partial \theta} \mu}_{V(\mu)}$$

Ex: $Y \sim \text{Poisson}(\lambda)$

$\mu = \lambda \qquad Var(Y) = \underbrace{1}_{\phi} \cdot \underbrace{\mu}_{v(\mu)}$

$\theta = \log \lambda \implies \mu = e^{\theta}$

$v(\mu) = \dfrac{\partial \mu}{\partial \theta} = e^{\theta} = \mu \implies v(\mu) = \mu$ ✓

Ex: $Y \sim \text{Bernoulli}(p)$

$\mu = p \qquad Var(Y) = \underbrace{1}_{\phi} \cdot \underbrace{\mu(1-\mu)}_{v(\mu)}$

$\theta = \log\left(\dfrac{p}{1-p}\right)$

$\implies \mu = \dfrac{e^{\theta}}{1+e^{\theta}}$

$v(\mu) = \dfrac{\partial \mu}{\partial \theta} = \dfrac{(1+e^{\theta})e^{\theta} - (e^{\theta})^2}{(1+e^{\theta})^2} = \dfrac{e^{\theta}}{(1+e^{\theta})^2} = \mu(1-\mu)$ ✓

Summarizing: $Y \sim EDM(\mu, \phi)$ $\mu = \mathbb{E}[Y]$

$\Rightarrow$ $f(y; \theta, \phi) = a(y, \phi) = \exp\left\{ \dfrac{y\theta - k(\theta)}{\phi} \right\}$

0) $\phi > 0$

where: 1) $\dfrac{\partial k(\theta)}{\partial \theta} = \mu$

2) $Var(Y) = \phi \dfrac{\partial^2 k(\theta)}{\partial \theta^2} = \phi \dfrac{\partial \mu}{\partial \theta} = \phi V(\mu)$

$Var(Y) > 0 \Rightarrow \dfrac{\partial \mu}{\partial \theta} > 0$

$\Rightarrow$ μ & θ are monotonic increasing functions of each other

3) $\theta = g(\mu)$, where $g(\mu)$ is monotonically increasing

GLM: $Y_i \sim EDM(\mu_i, \phi)$

$h(\mu_i) = \beta^T X_i$

Canonical link:

$h(\mu_i) = g(\mu_i) = \theta_i$ (canonical parameter)

Generalized linear models