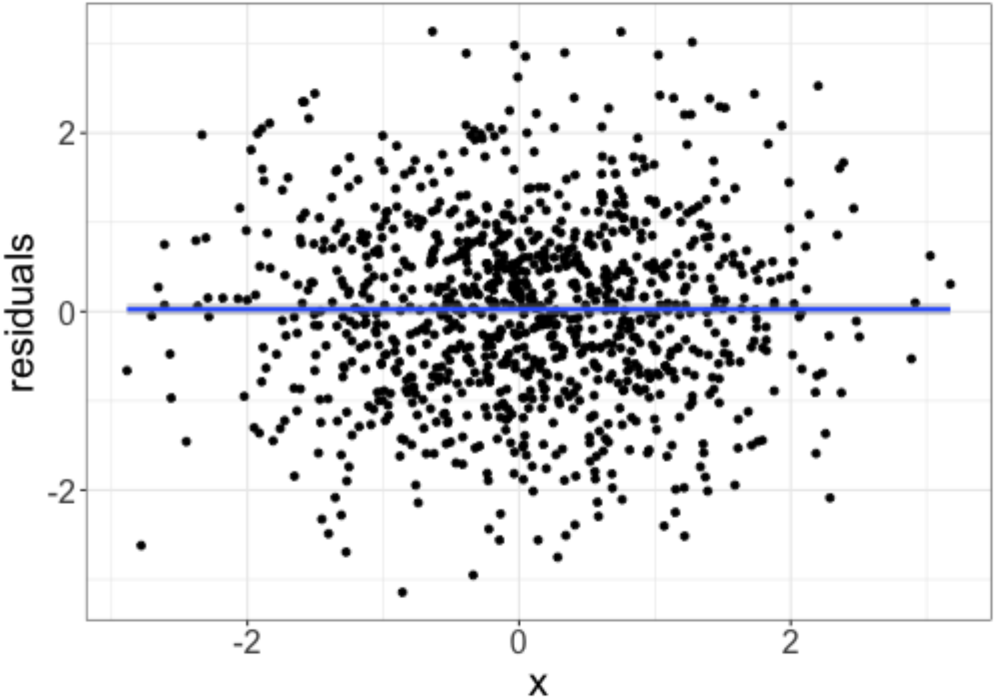
Logistic regression assumptions and diagnostics

- HW 1 due Tuesday
- Challenge 2 released on course website
  - Logistic regression "in the wild"

Last time: quantile residuals to assess shape



Warm up: Class activity, Part I

https://sta712-f22.github.io/class_activities/ca_lecture_6.html

- Generate data for which the logistic regression shape assumption doesn't hold
- See whether the violation shows up on a quantile residual plot

More logistic regression diagnostics

- Are there any outliers that could affect the fitted model?
- Are there issues with multicollinearity?

correlation
variance inflation
factors

leverage &
cook's distance

Leverage and Cook's Distance in linear regression

Linear regression: $Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_n \end{pmatrix}$ $\quad X = \begin{pmatrix} 1 & X_{11} & \cdots & X_{1k} \\ 1 & X_{21} & & \\ \vdots & \vdots & & \\ 1 & X_{n1} & \cdots & X_{nk} \end{pmatrix}$

$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, \sigma^2 I) \qquad \beta = \begin{bmatrix} \beta_0 \\ \vdots \\ \beta_k \end{bmatrix}$

$$\Rightarrow \hat{Y} = \underbrace{X(X^TX)^{-1}X^T}_{\text{"hat matrix" } H} Y$$

leverage of observation i (i-th diagonal element of H)

$\text{var}(Y - \hat{Y}) = \sigma^2 (I - H) \quad \Rightarrow \quad \text{Var}(Y_i - \hat{Y}_i) = \sigma^2 (1 - h_i)$

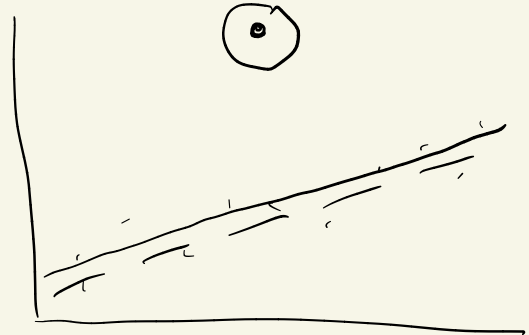
$h_i \leq 1 \qquad \uparrow h_i \Rightarrow \text{var}(Y_i - \hat{Y}_i) \downarrow$

SLR: $h_i = \dfrac{1}{n} + \dfrac{(X_i - \bar{X})^2}{\sum_j (X_j - \bar{X})^2}$

outlier, high leverage

↓

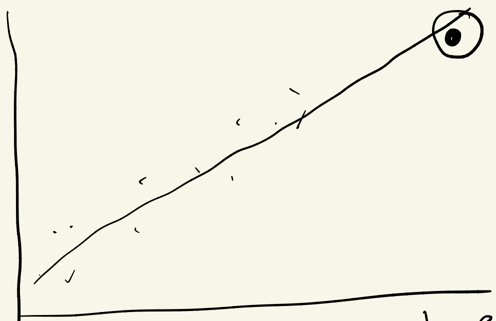
influential



outlier, low leverage

↓

not influential



not outlier, high leverage

=> not influential

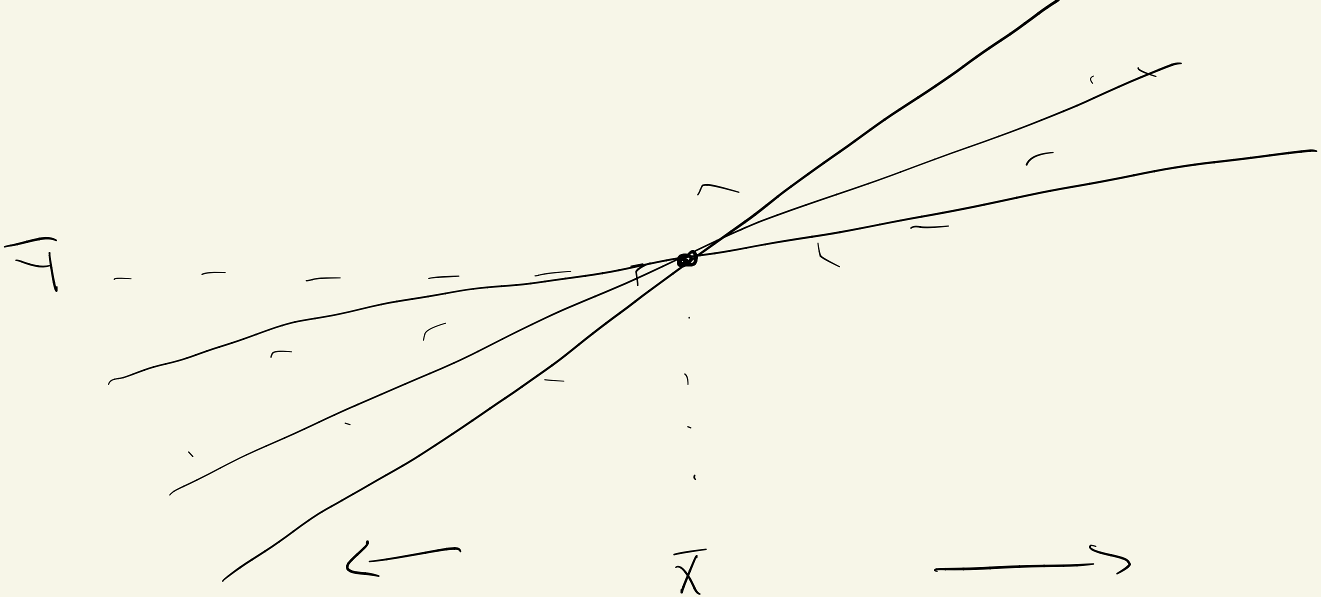
Cook's Distance

$$D_i = \frac{(y_i - \hat{y}_i)^2}{k\,\hat{\sigma}^2} \cdot \frac{h_i}{(1-h_i)^2}$$

Concerned when $D_i >$ threshold

e.g. 0.5

or 1



$\bar{Y}$
$\bar{x}$

Fisher scoring as Iteratively Reweighted Least Squares

Fisher scoring : $\beta^{(r+1)} = \beta^{(r)} + \mathcal{I}^{-1}(\beta^{(r)})\, u(\beta^{(r)})$

HW 1, #3 $\Rightarrow$ $u(\beta) = X^T(Y - p)$

$$p = \left[\underbrace{\frac{e^{\beta^T x_1}}{1+e^{\beta^T x_1}}}_{p_1}, \underbrace{\frac{e^{\beta^T x_2}}{1+e^{\beta^T x_2}}}_{p_2}, \ldots, \frac{e^{\beta^T x_n}}{1+e^{\beta^T x_n}} \right]^T$$

$\rightarrow$ $\mathcal{I}(\beta) = X^T W X$ $\quad W = \operatorname{diag}(p_1(1-p_1), p_2(1-p_2), \ldots, p_n(1-p_n))$

$$= \beta^{(r+1)} = \beta^{(r)} + (X^T W^{(r)} X)^{-1} X^T (Y - p^{(r)})$$

$$\beta^{(r+1)} = \beta^{(r)} + \left(X^T W^{(r)} X\right)^{-1} X^T (Y - p^{(r)})$$

$$= \underbrace{\left(X^T W^{(r)} X\right)^{-1} X^T W^{(r)} X}_{I} \beta^{(r)} + \left(X^T W^{(r)} X\right)^{-1} X^T (Y - p^{(r)})$$

$$= \left(X^T W^{(r)} X\right)^{-1} X^T W^{(r)} \underbrace{\left(X\beta^{(r)} + \left(W^{(r)}\right)^{-1} (Y - p^{(r)}) \right)}_{Z^{(r)}}$$

$\leftarrow$ <u>working</u> responses (at iteration r)

$$\Rightarrow \beta^{(r+1)} = \left(X^T W^{(r)} X\right)^{-1} X^T W^{(r)} Z^{(r)}$$

$\Rightarrow$ Fisher scoring is really weighted least squares with weights $W^{(r)}$ and response $Z^{(r)}$

Call this <u>iteratively reweighted least squares</u> (IRLS)

Weighted least squares

$$Y = X\beta + \varepsilon \qquad \varepsilon \sim N(0, W^{-1})$$

$$\Rightarrow \underbrace{W^{\frac{1}{2}} Y}_{Y_w} = \underbrace{W^{\frac{1}{2}} X}_{X_w} \beta + \underbrace{W^{\frac{1}{2}} \varepsilon}_{\varepsilon_w}$$

$$Var(W^{\frac{1}{2}}\varepsilon) = W^{\frac{1}{2}} Var(\varepsilon) W^{\frac{1}{2}} = I$$

$$\Rightarrow Y_w = X_w \beta + \varepsilon_w$$

Now solve by least squares:

$$\hat{\beta} = (X_w^T X_w)^{-1} X_w^T Y_w = (X^T W X)^{-1} X^T W Y$$

$$\hat{Y}_w = X_w (X_w^T X_w)^{-1} X_w^T Y_w = \underbrace{W^{\frac{1}{2}} X (X^T W X)^{-1} X^T W^{\frac{1}{2}}}_{\text{hat matrix } H} Y_w$$

Class Activity, Part II

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> Exploring leverage and Cook's distance!