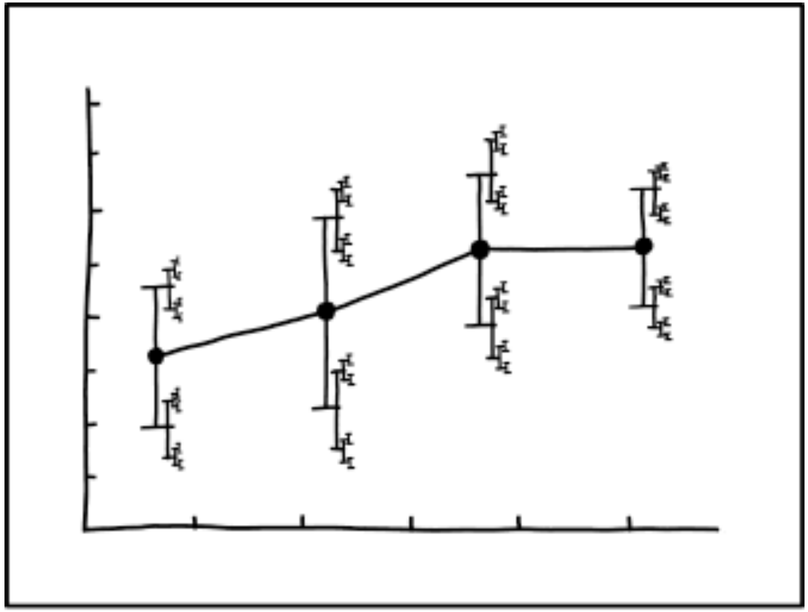
Asymptotic distribution of the MLE

HW 2 due date changed to next <u>Thursday</u> (9/15)

Warm-up

$$\frac{\bar{X}-\mu}{\hat{\sigma}/\sqrt{n}} \qquad \frac{\bar{X}-\mu}{s/\sqrt{n}}$$

$$\frac{\bar{X}-\mu}{s/\sqrt{n}} \approx N(0,1)$$



I DON'T KNOW HOW TO PROPAGATE ERROR CORRECTLY, SO I JUST PUT ERROR BARS ON ALL MY ERROR BARS.

Eventually:

$$\hat{\beta} \approx N(\beta, \mathcal{I}^{-1}(\beta))$$

> Why don't we estimate the standard error of our standard error?

Preliminaries

Let $X_1, X_2, X_3, \ldots$ be a sequence of random variables, and let X be another random variable. Let F_n be the CDF of X_n, and F the CDF of X.

Def: (Convergence in probability) $X_n \xrightarrow{p} X$ if

$\forall\, \varepsilon > 0$, $P(|X_n - X| > \varepsilon) \to 0$ as $n \to \infty$

Def: (convergence in distribution) $X_n \xrightarrow{d} X$ if,

$\forall\, t$ where F is continuous,

$F_n(t) \to F(t)$ as $n \to \infty$

Key results:

1) Weak law of large numbers (WLLN): If $X_1, \dots, X_n$, then

$$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} x_i \xrightarrow{P} \mathbb{E}[X_1]$$

2) Central limit theorem (CLT): Let $X_1, \dots, X_n$ be iid with mean μ and variance σ^2. Then

$\frac{\bar{X}-\mu}{s/\sqrt{n}}$

$$Z_n = \frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} Z \quad , Z \sim N(0,1)$$

3) Slutsky's theorem: If $X_n \xrightarrow{d} X$, $Y_n \xrightarrow{P} c \in \mathbb{R}$, then $X_n Y_n \xrightarrow{d} cX$

Fact: $\hat{\sigma} \xrightarrow{P} \sigma$

Asymptotic distribution of the MLE

Suppose that $Y_1, \ldots, Y_n$ are iid w/ probability function $f(Y_i; \theta)$, $\theta \in \mathbb{R}$

Let $\ell(\theta) = \sum_{i=1}^{n} \log f(Y_i; \theta)$, $\hat{\theta}$ = MLE of θ

$$\mathcal{I}_1(\theta) = -\mathbb{E}\left[\frac{d^2}{d\theta^2} \log f(Y;\theta)\right]$$

$$\boxed{\underline{\text{Theorem}}: \quad \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0, \mathcal{I}_1^{-1}(\theta))}$$

<u>Proof sketch</u>: 1) $\sqrt{n}(\hat{\theta} - \theta) \approx \dfrac{-\frac{1}{\sqrt{n}} \ell'(\theta)}{\frac{1}{n} \ell''(\theta)}$

2) $\frac{1}{n} \ell''(\theta) \xrightarrow{P} -\mathcal{I}_1(\theta)$

3) $\frac{1}{\sqrt{n}} \ell'(\theta) \xrightarrow{d} N(0, \mathcal{I}_1(\theta))$

4) Slutsky's theorem: $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \overbrace{\frac{1}{\mathcal{I}_1(\theta)} N(0, \mathcal{I}_1(\theta))}^{N(0, \frac{1}{\mathcal{I}_1(\theta)})}$

Proof: $\hat{\theta} = \text{MLE} \Rightarrow \ell'(\hat{\theta}) = 0$

If $\hat{\theta} \approx \theta$ ("close enough")

$$0 = \ell'(\hat{\theta}) \approx \ell'(\theta) + (\hat{\theta} - \theta)\,\ell''(\theta)$$ (First-order Taylor expansion)

$$\Rightarrow \hat{\theta} - \theta \approx -\frac{\ell'(\theta)}{\ell''(\theta)}$$

$$\Rightarrow \sqrt{n}(\hat{\theta} - \theta) \approx \sqrt{n}\left(-\frac{\ell'(\theta)}{\ell''(\theta)}\right)$$

$$= -\frac{\frac{1}{\sqrt{n}}\ell'(\theta)}{\frac{1}{n}\ell''(\theta)}$$

Denominator $\quad \frac{1}{n} l''(\theta) = \frac{1}{n} \sum_{i=1}^{n} l''_i(\theta)$, $l''_i = \frac{\partial^2}{d\theta^2} \log f(Y_i;\theta)$

By WLLN , $\frac{1}{n} \sum_{i=1}^{n} l''_i(\theta) \xrightarrow{P} \mathbb{E}[l''_i(\theta)]$

and we know that $\mathcal{I}_1(\theta) = -\mathbb{E}\left[\frac{\partial^2}{d\theta^2} \log f(Y_i;\theta)\right]$

$\Rightarrow \quad \frac{1}{n} \sum_{i=1}^{n} l''_i(\theta) \xrightarrow{P} -\mathcal{I}_1(\theta)$

$\Rightarrow \quad \frac{1}{n} l''(\theta) \xrightarrow{P} -\mathcal{I}_1(\theta)$

Numerator: $\frac{1}{\sqrt{n}} l'(\theta)$

$$\mathbb{E}\left[\frac{1}{\sqrt{n}} l'(\theta)\right] = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right]$$

$Y_1, \dots, Y_n$ are iid, $\Rightarrow$ $\frac{1}{\sqrt{n}} \cdot n \cdot \mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right]$

$$= \sqrt{n}\, \mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right]$$

Claim: $\mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right] = 0$

Pf: $\mathbb{E}\left[\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right] = \int \frac{\partial}{\partial\theta} \log f(y;\theta)\, f(y;\theta)\, d\theta$

$$= \int \frac{1}{\cancel{f(y;\theta)}} \cdot \frac{\partial}{\partial\theta} f(y;\theta) \cdot \cancel{f(y;\theta)}\, d\theta = \int \frac{\partial}{\partial\theta} f(y;\theta)\, d\theta$$

$$= \frac{\partial}{\partial\theta} \int f(y;\theta)\, d\theta = 0 \quad //$$

<u>Numerator</u>: $\text{Var}\left(\frac{1}{\sqrt{n}} \ell'(\theta)\right) = \frac{1}{n} \cdot \sum_{i=1}^{n} \text{Var}\left(\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right)$

$$= \text{Var}\left(\frac{\partial}{\partial\theta} \log f(Y_i;\theta)\right)$$